Mathematical Finance
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Recall 2

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a filtration $\mathbb{F}$.

Continuous-time martingales

1. When is an $\mathbb{R}$-valued process $M := (M_t)_{t\in[0,T]}$ said to be an $(\mathbb{F},\mathbb{P})$-super-martingale? And an $(\mathbb{F},\mathbb{P})$-sub-martingale?
2. Let X be a right-continuous $(\mathbb{F},\mathbb{P})$-super-martingale. Which are some inequalities satisfied by X?
3. Let X be an $\mathbb{F}$-adapted process. Can you state one condition under which X has a $\mathbb{P}$-modification Y that has left- and right-limits everywhere, and there is a countable subset $S \subset [0,\infty)$ for which Y_t is right-continuous at every $t \notin S$?
4. Let X be an $\mathbb{F}$-adapted process. Can you give some conditions under which X has a *càdlàg* $\mathbb{P}$-modification?
5. Let $\mathbb{F}$ be right-continuous. When does an $(\mathbb{F},\mathbb{P})$-martingale have a *càdlàg* $\mathbb{P}$-modification? When does an $(\mathbb{F},\mathbb{P})$-sub-martingale have a *càdlàg* $\mathbb{P}$-modification?

Convergence results

1. Can you give a convergence result when
   (a) X is a non-negative right-continuous $(\mathbb{F},\mathbb{P})$-super-martingale?
   (b) X is a $\mathbb{P}$-uniformly integrable $(\mathbb{F},\mathbb{P})$-super-martingale?
   (c) X is a $\mathbb{P}$-uniformly integrable $(\mathbb{F},\mathbb{P})$-martingale?
2. Can you state a convergence result when $(X^n)_{n\in\mathbb{N}}$ is a sequence of *càdlàg* $(\mathbb{F},\mathbb{P})$-martingales? Can you prove it?

Martingales and simple process

1. Let X be an $\mathbb{R}$-valued process and ξ an $\mathbb{F}$-predictable simple process bounded by 1. Can you state the inequality satisfied by $\int_0^t \xi_s \mathrm{d}X_s$ when
   (a) X is an $(\mathbb{F},\mathbb{P})$-martingale such that $X_t \in \mathbb{L}^2(\mathbb{R},\mathbb{F},\mathbb{P})$ for any $t \geq 0$?
   (b) X is an $(\mathbb{F},\mathbb{P})$-martingale?
2. Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of *càdlàg* $(\mathbb{F},\mathbb{P})$-martingales, and X a process such that $E^{\mathbb{P}}[|X^n_t - X_t|] \to 0$, as $n \to \infty$, what can you say about X? Can you prove it?

Everything local

1. What is an $(\mathbb{F},\mathbb{P})$-local martingale?
2. Can you show that any $(\mathbb{F},\mathbb{P})$-local martingale is $(\mathbb{F},\mathbb{P})$-locally integrable?
3. Can you prove that a non-negative $(\mathbb{F},\mathbb{P})$-local super-martingale X which is such that in addition X_0 is $\mathbb{P}$-integrable is an $(\mathbb{F},\mathbb{P})$-super-martingale?

The Doob-Meyer decomposition

1. Let X be measurable stochastic process. What does it mean that X is of class $(\mathbb{F},\mathbb{P})$-(D)? And of class $(\mathbb{F},\mathbb{P})$-(DL)?
2. Let X be a *càdlàg* $(\mathbb{F},\mathbb{P})$-martingale. When is X of class $(\mathbb{F},\mathbb{P})$-(DL)? When is X of class $(\mathbb{F},\mathbb{P})$-(D)?